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Numerical Approach to Computing Nonlinear H_∞ Control Laws

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An algorithm and data structure are developed for finding the Taylor series solution of the Hamilton–Jacobi–Isaacs equation associated with the nonlinear H_∞ control problem. This algorithm yields a set of linear algebraic equations that not only lead to a transparent solvability condition of the Hamilton–Jacobi–Isaacs equation in the form of the Taylor series, but also furnish a systematic procedure to generate the coefficients of the Taylor series. This algorithm is illustrated through a missile pitch autopilot design example.

Nomenclature

C_m = aerodynamic coefficient (moment)
C_z = aerodynamic coefficient (force)
$\bar{c}$ = reference length
I_{yy} = moment of inertia
M_m = Mach number
m = missile mass
Q = pitch rate
$\bar{q}$ = dynamic pressure
S = reference area
α = angle of attack
δ_e = pitch control

Introduction

RECENTLY, there have been considerable efforts in developing nonlinear H_∞ control theory (see Refs. 1–5). The typical problem addressed by this approach is the design of feedback laws that internally stabilize a nonlinear plant and attenuate the influence of exogenous inputs such as disturbances and reference commands. Following the treatment in Refs. 4 and 5, we can describe the (state feedback) nonlinear H_∞ control problem as follows:

Consider the following multi-input, multi-output nonlinear systems:

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h_1(x) + k_{12}(x)u \end{aligned} \tag{1}$$

where $x \in R^n$ is the plant state, $u \in R^{m_2}$ is the plant input, $w \in R^{m_1}$ is a set of exogenous input variables, and $z \in R^p$ is a penalty variable. It is assumed that all functions involved in this setup are smooth and defined in a neighborhood U of the origin in R^n and have zero values at the origin. To simplify the derivation of the controller, we also make the following assumption on the plant:

$$h_1^T(x)k_{12}(x) = 0, \qquad k_{12}^T(x)k_{12}(x) = R_2 \tag{2}$$

with R_2 a nonsingular constant matrix.

The feedback control law takes the form

$$u = l(x) \tag{3}$$

where $l(x)$ is a locally defined sufficiently smooth function satisfying $l(0) = 0$.

The purpose of control is twofold: to achieve closed-loop stability and to attenuate the effect of the disturbance input w to the penalty variable z. Here closed-loop stability means that the equilibrium $x = 0$ of the closed-loop system is stable. The disturbance attenuation is characterized in the following way: Given a real number $0 < \gamma$, it is said that the exogenous signals are locally attenuated by γ if there exists a neighborhood U of the point $x = 0$ such that, for every $T > 0$ and for every piecewise continuous function w: $[0, T] \to R^{m_1}$ for which the state trajectory of the closed-loop system starting from $x(0) = 0$ remains in U for all $t \in [0, T]$, the response z: $[0, T] \to R^p$ of Eqs. (1) and (3) satisfies

$$\int_0^T z^T(s)z(s)\,\mathrm{d}s \le \gamma^2 \int_0^T w^T(s)w(s)\,\mathrm{d}s \tag{4}$$

The Hamiltonian function associated with the above problem is

$$\begin{aligned} \boldsymbol{H}(x, p, w, u) &= p^T(f(x) + g_1(x)w + g_2(x)u) \\ &\quad + \tfrac{1}{2}\left(\|h_1(x) + k_{12}(x)u)\|^2 - \gamma^2\|w\|^2\right) \end{aligned} \tag{5}$$

and under assumption (2), Eq. (5) can be rewritten as

$$\begin{aligned} \boldsymbol{H}(x, p, w, u) &= p^T f(x) + \tfrac{1}{2}h_1^T(x)h_1(x) \\ &\quad + \begin{bmatrix} p^T g_1 & p^T g_2 \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} + \frac{1}{2}\begin{bmatrix} w \\ u \end{bmatrix}^T R \begin{bmatrix} w \\ u \end{bmatrix} \end{aligned} \tag{6}$$

where

$$R = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & R_2 \end{bmatrix}$$

The Hamilton–Jacobi–Isaacs equation in Refs. 3 and 5 is given as

$$H(x, p) \overset{\text{def}}{=} \boldsymbol{H}(x, p, \alpha_1(x, p), \alpha_2(x, p)) = 0 \tag{7}$$

where

$$\alpha_1(x, p) = \frac{1}{\gamma^2}g_1^T p, \qquad \alpha_2(x, p) = -R_2^{-1}g_2^T p \tag{8}$$

It was shown in Refs. 2–4 that if there exists a positive-definite C^1 function $V(x)$ locally defined in a neighborhood of the origin in R^n that satisfies

$$\boldsymbol{H}\left(x, V_x^T, \alpha_1\left(x, V_x^T\right), \alpha_2\left(x, V_x^T\right)\right) = 0 \tag{9}$$

or, more explicitly,

$$V_x f(x) + \frac{1}{2}h_1^T h_1 + \frac{1}{2}V_x\left(\frac{g_1 g_1^T}{\gamma^2} - g_2 R_2^{-1} g_2\right)V_x^T = 0 \tag{10}$$

where V_x denotes the Jacobian matrix of $V(x)$, then the state feedback control law given by

$$u = -R_2^{-1}(x)g_2^T(x)V_x^T \tag{11}$$

achieves disturbance attenuation with performance level specified by γ. Figure 1 shows the closed-loop system under the state feedback nonlinear H_∞ control law.

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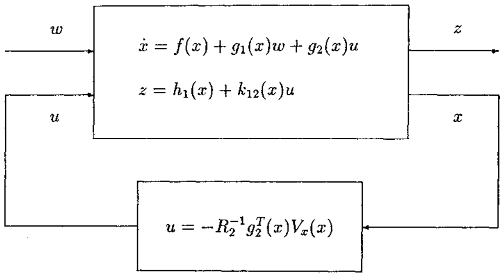

Fig. 1 State feedback nonlinear H_∞ control design diagram.

Due to the nonlinear nature, it is rarely possible to find a closed-form solution for the Hamilton–Jacobi–Isaacs equation. An approximation approach to solving Eq. (10) via the Taylor series was suggested in Refs. 4 and 6. In particular, Kang et al.[6] gave the necessary and sufficient conditions for the existence of the Taylor series solution of the Hamilton–Jacobi–Isaacs equation. Their result is summarized below. Expand $H(x, p)$ as

$$H(x,p) = \tfrac{1}{2}x^T H_{xx}x + p^T H_{px}x + \tfrac{1}{2}p^T H_{pp}p + H^{[3+]}(x,p) \quad (12)$$

where $H^{[3+]}(x, p)$ consists of cubic and higher order terms of $H(x, p)$. Then the solution of Eq. (9) can be written as

$$V(x) = \tfrac{1}{2}x^T Px + V^{[3+]}(x) \quad (13)$$

where P satisfies

$$\tfrac{1}{2}x^T H_{xx}x + x^T PH_{px}x + \tfrac{1}{2}x^T PH_{pp}Px = 0 \quad (14)$$

and $V^{[3+]}(x)$, cubic and higher order terms of $V(x)$, satisfies

$$V_x^{[3+]}(H_{px} + H_{pp}P)x = -\tfrac{1}{2}V_x^{[3+]}H_{pp}\left(V_x^{[3+]}\right)^T - H^{[3+]}(x, V_x) \quad (15)$$

In Eq. (15), $V_x^{[3+]}$ denotes the Jacobian matrix of $V^{[3+]}(x)$. It is clear that P can be obtained by solving the algebraic Riccati equation

$$H_{px}^T P + PH_{px} + PH_{pp}Px + H_{xx} = 0 \quad (16)$$

Kang et al.[6] also showed that the coefficient vectors of the power series solution $V^{[3+]}(x)$ of Eq. (15) are governed by a sequence of linear mappings and gave conditions for these mappings to be both surjective and injective. However, no specific algorithm is available for generating these coefficient vectors.

The objective of this paper is to provide an algorithm and data structure that give the Taylor series solution of Eq. (15). This algorithm yields a sequence of explicit linear algebraic equations that are representations of the linear mappings derived in Ref. 6. This set of equations not only leads to a transparent proof of the existence condition of the Taylor series solution similar to that discovered in Ref. 6, but also furnishes a systematic procedure to generate the coefficient vectors of the Taylor series expansion of $V^{[3+]}$. A missile pitch autopilot design is used to illustrate the algorithm and data structure.

Power Series Solution of Hamilton–Jacobi–Isaacs Equation

Our approach is much the same as that used in Ref. 7. Therefore, the following notation introduced in Ref. 7 will be given first: For any matrix K, we define

$$K^{(0)} = 1, \qquad K^{(1)} = K$$

$$K^{(i)} = \underbrace{K \otimes K \otimes \cdots \otimes K}_{i\,\text{factors}} \qquad i = 2, 3, \ldots$$

where $\otimes$ stands for the Kronecker product. Also for any n-dimensional vector $x = [x_1, \ldots, x_n]^T$, we define

$$x^{[0]} = 1, \qquad x^{[1]} = x$$

$$x^{[k]} = \left[x_1^k \quad x_1^{k-1}x_2 \quad \cdots \quad x_1^{k-1}x_n \quad x_1^{k-2}x_2^2 \quad x_1^{k-2}x_2x_3 \quad \cdots \quad x_1^{k-2}x_2x_n \quad \cdots \quad x_n^k\right]^T \qquad k \geq 1$$

It is clear that there exist constant matrices M_k and N_k such that

$$x^{[k]} = M_k x^{(k)}, \qquad x^{(k)} = N_k x^{[k]}$$

For example, with $n = 2$, $x^{(2)}$ and $x^{[2]}$ are given by

$$x^{(2)} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2x_1 \\ x_2^2 \end{bmatrix}, \qquad x^{[2]} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$

Correspondingly, M_2 and N_2 are given by

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the above notation gives the following unique expression for $V(x)$:

$$V(x) = \frac{1}{2}x^T Px + \sum_{k=3}^{\infty} P_k x^{[k]} \stackrel{\text{def}}{=} \frac{1}{2}x^T Px + V^{[3+]}(x) \quad (17)$$

Our purpose is to derive explicit equations that generate all row vectors $P_k, k = 3, 4, \ldots$. To this end, we first list some useful identities involving the Kronecker product as follows:

Lemma

1) For any matrices A, B, C, D of appropriate dimensions,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \quad (18)$$

2) For $k \geq 1$,

$$\frac{\partial x^{(k)}}{\partial x} = \sum_{i=1}^{k} x^{(i-1)} \otimes I_n \otimes x^{(k-i)} \quad (19)$$

where I_n denotes an $n \times n$ identity matrix.

3) For $x \in R^n$, $y \in R^m$, and $A \in R^{n\times m}$,

$$x^T Ay = \text{row}(A)(x \otimes y) \quad (20)$$

where row $R^{n\times m} \to R^{1\times nm}$ is an operator that maps the $n \times m$ matrix $A = [a_{ij}]$ to a $1 \times mn$ row vector row(A) in the following way:

$$\text{row}(A) = [a_{11}a_{12}\cdots a_{1m}\cdots a_{n1}\cdots a_{nm}]$$

4) For any integers $i, j, k \geq 0$ and matrix T of dimension $n \times n^k$,

$$\left(x^{(i)} \otimes I_n \otimes x^{(j)}\right)T_x^{(k)} = \left(I_n^{(i)} \otimes T \otimes I_n^{(j)}\right)x^{(i+j+k)} \quad (21)$$

Proof. Equations (18–20) follow straightforwardly from the definition of the Kronecker product. Equation (21) can be proved as follows:

$$\begin{aligned}\left(x^{(i)} \otimes I_n \otimes x^{(j)}\right)Tx^{(k)} &= \left[x^{(i)} \otimes \left(I_n \otimes x^{(j)}\right)\right][1 \otimes (Tx^{(k)} \otimes 1)] \\ &= x^{(i)} \otimes \left(I_n \otimes x^{(j)}\right)(Tx^{(k)} \otimes 1) \\ &= x^{(i)} \otimes (Tx^{(k)} \otimes x^{(j)}) \\ &= \left(I_n^{(i)}x^{(i)}\right) \otimes (Tx^{(k)}) \otimes x^{(j)} \\ &= \left(I_n^{(i)} \otimes T\right)x^{(i+k)} \otimes x^{(j)} \\ &= \left(I_n^{(i)} \otimes T\right)x^{(i+k)} \otimes \left(I_n^{(j)}x^{(j)}\right) \\ &= \left(I_n^{(i)} \otimes T \otimes I_n^{(j)}\right)x^{(i+j+k)}\end{aligned}$$

Note that in deriving Eq. (21), we have repeatedly utilized identity (18). With these preparations, we are ready to state our major result of this section:

Theorem. The coefficient vectors P_k of Eq. (17) satisfies the equation

$$P_k U_k = V_k \tag{22}$$

where

$$U_k = M_k \sum_{i=1}^{k} I_n^{(i-1)} \otimes (H_{px} + H_{pp}P) \otimes I_n^{(k-i)} N_k \tag{23}$$

and V_k depends only on $P, P_3, \ldots, P_{k-1}$. Furthermore, the eigenvalues of U_k are given by

$$\lambda = \lambda_{i_1} + \cdots + \lambda_{i_k} \tag{24}$$

where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$ and $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $H_{px} + H_{pp}P$.

Proof. By definition,

$$V_x^{[3+]} = \frac{\partial\left(\sum_{k=3}^{\infty} P_k x^{[k]}\right)}{\partial x} = \frac{\partial\left(\sum_{k=3}^{\infty} P_k M_k x^{(k)}\right)}{\partial x}$$

Substituting Eq. (19) in the above equation gives

$$V_x^{[3+]}(x) = \sum_{k=3}^{\infty} P_k M_k \sum_{i=1}^{k} x^{(i-1)} \otimes I_n \otimes x^{(k-i)} \tag{25}$$

Substituting Eq. (25) into the left side of Eq. (15) and using Eq. (21) gives

$$\begin{aligned} & V_x^{[3+]}(H_{px} + H_{pp}P)x \\ &= \sum_{k=3}^{\infty} P_k M_k \sum_{i=1}^{k} \left(x^{(i-1)} \otimes I_n \otimes x^{(k-i)}\right)(H_{px} + H_{pp}P)x \\ &= \sum_{k=3}^{\infty} P_k M_k \left(\sum_{i=1}^{k} I_n^{(i-1)} \otimes (H_{px} + H_{pp}P) \otimes I_n^{(k-i)}\right) x^{(k)} \\ &= \sum_{k=3}^{\infty} P_k U_k x^{[k]} \end{aligned} \tag{26}$$

On the other hand, by inspection, it is not difficult to find that the right-hand side of Eq. (15) can be arranged as

$$-\frac{1}{2} V_x^{[3+]} H_{pp} \left(V_x^{[3+]}\right)^T - H^{[3+]}(x, V_x) = \sum_{k=3}^{\infty} V_k x^{[k]} \tag{27}$$

where V_k depends only on $P, P_3, \ldots, P_{k-1}$. In fact, it is possible to explicitly express V_k in terms of $P, P_3, \ldots, P_{k-1}$, as will be done later.

To establish Eq. (24), we only note that if $x(t)$ satisfies

$$\dot{x}(t) = (H_{px} + H_{pp}P)x(t) \tag{28}$$

then $x^{[k]}$ satisfies

$$\dot{x}^{[k]}(t) = U_k x^{[k]}(t) \tag{29}$$

Therefore, the eigenvalues of U_k are given by Eq. (24). In fact, this argument has also been used in Refs. 7 and 8.

The above theorem immediately leads to the existence condition of the Taylor series solution of Eq. (15) as follows:

Corollary. There exists a unique solution P_k of Eq. (22) for $k \geq 3$ if and only if, for all $k \geq 3$ all $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$,

$$\lambda_{i_1} + \cdots + \lambda_{i_k} \neq 0 \tag{30}$$

Remark. The same condition as Eq. (30) is established in Ref. 6 using a quite different argument.

Remark. If we denote the Taylor series expansion of functions $f(x)$, $g_1(x)$, $g_2(x)$, and $h(x)$ as

$$f(x) = Ax + f^{[2+]}(x), \qquad g_1(x) = B_1 + g_1^{[1+]}(x)$$

$$g_2(x) = B_2 + g_2^{[1+]}(x), \qquad h_1(x) = C_1 x + h^{[2+]}(x)$$

then it is easy to identify that

$$H_{px} = A, \qquad H_{xx} = C_1^T C_1$$

$$H_{pp} = \frac{B_1 B_1^T}{\gamma^2} - B_2 R_2^{-1} B_2^T$$

Therefore $H_{px} + H_{pp}P$ corresponds to the closed-loop system matrix of the linearized plant under linear state feedback H_∞ control. As a result, condition (30) always holds as long as the standard linear state feedback H_∞ suboptimal control for the linearized plant corresponding to γ is solvable (see Ref. 9).

Explicit Expression of V_k

Now we turn to the problem of deriving an explicit expression for V_k. First note that there exists a constant matrix S_k of dimension n^{k-1} by n determined by P_k such that

$$P_k M_k \sum_{i=1}^{k} x^{(i-1)} \otimes I_n \otimes x^{(k-i)} = (x^T)^{(k-1)} S_k \tag{31}$$

In fact, we have

$$x^{(i-1)} \otimes I_n \otimes x^{(k-i)}$$

$$= \begin{bmatrix} x_1^{(i-1)} x^{(k-i)} & 0 & \cdots & 0 \\ 0 & x_1^{(i-1)} x^{(k-i)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_1^{(i-1)} x^{(k-i)} \\ x_1^{(i-2)} x_2 x^{(k-i)} & 0 & \cdots & 0 \\ 0 & x_1^{(i-2)} x_2 x^{(k-i)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_1^{(i-2)} x_2 x^{(k-i)} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(i-1)} x^{(k-i)} & 0 & \cdots & 0 \\ 0 & x_n^{(i-1)} x^{(k-i)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n^{(i-1)} x^{(k-i)} \end{bmatrix}$$

Therefore, if we equally partition $P_k M_k$ as a $1 \times n^i$-block matrix as

$$P_k M_k = \Big[\underbrace{P_{1\cdots 11}}_{i\text{-tuple}} \cdots \underbrace{P_{1\cdots 1n}}_{i\text{-tuple}} \underbrace{P_{1\cdots 21}}_{i\text{-tuple}} \cdots \underbrace{P_{1\cdots 2n}}_{i\text{-tuple}} \cdots$$

$$\cdots \underbrace{P_{n\cdots n1}}_{i\text{-tuple}} \cdots \underbrace{P_{n\cdots nn}}_{i\text{-tuple}}\Big]$$

then it holds that

$$P_k M_k x^{(i-1)} \otimes I_n \otimes x^{(k-i)} = \left(P_k^i x^{(k-1)}\right)^T \tag{32}$$

where

$$P_k^i = \begin{bmatrix} \underbrace{P_{1\cdots 11}}_{i\text{-tuple}} & \underbrace{P_{1\cdots 21}}_{i\text{-tuple}} & \cdots & \underbrace{P_{n\cdots n1}}_{i\text{-tuple}} \\ \underbrace{P_{1\cdots 12}}_{i\text{-tuple}} & \underbrace{P_{1\cdots 22}}_{i\text{-tuple}} & \cdots & \underbrace{P_{n\cdots n2}}_{i\text{-tuple}} \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{P_{1\cdots 1n}}_{i\text{-tuple}} & \underbrace{P_{1\cdots 2n}}_{i\text{-tuple}} & \cdots & \underbrace{P_{n\cdots nn}}_{i\text{-tuple}} \end{bmatrix}$$

Clearly S_k is given by

$$S_k = \sum_{i=1}^{k} \left(P_k^i\right)^T \qquad k = 3, 4, \ldots \tag{33}$$

Remark. As a result of Eq. (31), we have

$$V_x^T = \sum_{k=2}^{\infty} S_k^T x^{(k-1)}$$

where $S_2 = P$ and S_k is given by Eq. (33). Moreover, the control law (11) can be approximated by

$$u = -R_2^{-1} g_2^T(x)\left(Px + \sum_{k=3}^{\infty} S_k^T N_{k-1} x^{[k-1]}\right)$$

Now we can proceed as follows:

$$\begin{aligned}
&V_x^{[3+1]} H_{pp}\left(V_x^{[3+1]}\right)^T \\
&= \sum_{k=3}^{\infty} P_k M_k \left(\sum_{i=1}^{k} x^{(i-1)} \otimes I_n \otimes x^{(k-i)}\right) \\
&\quad \times H_{pp}\left(\sum_{k=3}^{\infty} S_k^T x^{(k-1)}\right) \\
&= \sum_{k=4}^{\infty} \sum_{\substack{l+m=k+2\\ l,m\geq 3}} P_l M_l \left(\sum_{i=1}^{l} x^{(i-1)} \otimes I_n \otimes x^{(l-i)}\right) \\
&\quad \times H_{pp} S_m^T x^{(m-1)} \\
&= \sum_{k=4}^{\infty}\left[\sum_{\substack{l+m=k+2\\ l,m\geq 3}} P_l M_l \left(\sum_{i=1}^{l} I_n^{(i-1)} \otimes H_{pp} S_m^T \otimes I_n^{(l-i)}\right)\right] x^{(k)} \\
&= \sum_{k=4}^{\infty} Z_k x^{(k)}
\end{aligned} \tag{34}$$

where

$$Z_k = \left[\sum_{\substack{l+m=k+2\\ l,m\geq 3}} P_l M_l \left(\sum_{i=1}^{l} I_n^{(i-1)} \otimes H_{pp} S_m^T \otimes I_n^{(l-i)}\right)\right] \qquad k = 4, 5, \ldots \tag{35}$$

Next we note that

$$\begin{aligned}
H^{[3+1]}(x,p) &= p^T(f(x) - Ax) + \frac{1}{2}\left(h_1^T h_1 - x^T C_1^T C_1 x\right) \\
&\quad + \frac{1}{2} p^T\left(\frac{g_1 g_1^T - B_1 B_1^T}{\gamma^2} - \left(g_2 R_2^{-1} g_2 - B_2 R_2^{-1} B_2^T\right)\right) p
\end{aligned} \tag{36}$$

Now let

$$\begin{aligned}
&f(x) = \sum_{m=1}^{\infty} A_m x^{(m)}, \qquad h(x) = \sum_{m=1}^{\infty} C_m x^{(m)} \\
&g_1(x) = \left[g_{11}(x) \cdots g_{1m_1}(x)\right], \qquad g_2(x) = \left[g_{21}(x) \cdots g_{2m_2}(x)\right] \\
&g_{1l}(x) = \sum_{m=0}^{\infty} B_{1l}^m x^{(m)}, \qquad g_{2l}(x) = \sum_{m=0}^{\infty} B_{2l}^m x^{(m)}
\end{aligned} \tag{37}$$

Then

$$\begin{aligned}
V_x(f(x) - Ax) &= \left(\sum_{l=2}^{\infty} (x^{(l-1)})^T S_l\right)\left(\sum_{m=2}^{\infty} A_m x^{(m)}\right) \\
&= \sum_{k=3}^{\infty} \sum_{\substack{l+m=k+1\\ l,m\geq 2}} (x^{(l-1)})^T S_l A_m x^{(m)} \\
&= \sum_{k=3}^{\infty} E_k x^{(k)}
\end{aligned} \tag{38}$$

where

$$E_k = \sum_{\substack{l+m=k+1\\ l,m\geq 2}} \mathrm{row}(S_l A_m) \qquad k = 3, 4, \ldots \tag{39}$$

$$\begin{aligned}
h_1(x)^T h_1(x) &= \left(\sum_{l=1}^{\infty} C_l x^{(l)}\right)^T \sum_{m=1}^{\infty} C_m x^{(m)} \\
&= \sum_{k=2}^{\infty} \sum_{\substack{l+m=k\\ l,m\geq 1}} (x^{(l)})^T C_l^T C_m x^{(m)} \\
&= \sum_{k=2}^{\infty} F_k x^{(k)}
\end{aligned} \tag{40}$$

where

$$F_k = \sum_{\substack{l+m=k\\ l,m\geq 1}} \mathrm{row}\left(C_l^T C_m\right) \qquad k = 2, 3, \ldots \tag{41}$$

Also for $i = 1, 2,\ j = 1, \ldots, m_i$,

$$\begin{aligned}
V_x g_{ij}(x) &= \left(\sum_{l=2}^{\infty} (x^{(l-1)})^T S_l\right)\left(\sum_{m=0}^{\infty} B_{ij}^m x^{(m)}\right) \\
&= \sum_{k=1}^{\infty} \sum_{\substack{l+m=k+1\\ l\geq 2, m\geq 0}} (x^{(l-1)})^T S_l B_{ij}^m x^{(m)} \\
&= \sum_{k=1}^{\infty} W_{ij}^k x^{(k)}
\end{aligned} \tag{42}$$

where

$$W_{ij}^k = \sum_{\substack{l+m=k+1\\ l\geq 2, m\geq 0}} \mathrm{row}\left(S_l B_{ij}^m\right) \qquad k = 1, 2, \ldots \tag{43}$$

Therefore,

$$V_x \frac{g_1 g_1^T}{\gamma^2} V_x^T = \sum_{j=1}^{m_1} V_x \frac{g_{1j} g_{1j}^T}{\gamma^2} V_x^T = G_k x^{(k)} \tag{44}$$

where

$$G_k = \sum_{\substack{l+m=k\\ l,m\geq 1}} \sum_{j=1}^{m_1} \frac{\mathrm{row}\left[\left(W_{1j}^l\right)^T W_{1j}^m\right]}{\gamma^2} \qquad k = 2, 3, \ldots \tag{45}$$

Similarly,

$$V_x g_2 R_2^{-1} g_i^T V_x^T = \sum_{j=1}^{m_2} V_x g_{2j} R_2^{-1} g_{2j}^T V_x^T = H_k x^{(k)} \tag{46}$$

where

$$H_k = \sum_{\substack{l+m=k \\ l,m\geq 1}} \sum_{j=1}^{m_2} \text{row}\left[\left(W_{2j}^l\right)^T R_2^{-1} W_{2j}^m\right] \qquad k = 2, 3, \ldots \qquad (47)$$

Substituting Eqs. (38), (40), (44), and (46) into Eq. (36) gives

$$H^{[3+]}(x, V_x) = \sum_{k=3}^{\infty} \left(E_k + \frac{F_k + G_k - H_k}{2}\right) N_k x^{[k]} \qquad (48)$$

Finally, substituting Eqs. (34) and (48) into Eq. (27) gives

$$V_3 = -\left[E_3 + \tfrac{1}{2}(F_3 + G_3 - H_3)\right] N_3$$

$$V_k = -\tfrac{1}{2}(Z_k + 2E_k + F_k + G_k - H_k) N_k \qquad k = 4, 5, \ldots \qquad (49)$$

It is clear from the explicit expressions of Z_k, E_k, F_k, G_k, and H_k that V_k depends only on $P, P_3, \ldots, P_{k-1}$.

Missile Autopilot Design Example

So far there have been no clear guidelines in formulating a nonlinear control problem into a nonlinear H_∞ minimization problem. Issues such as uncertainty modeling and weighting selection are yet to be pursued. Here, we will formulate a missile's pitch autopilot design as a nonlinear H_∞ control problem. The emphasis is placed, however, on illustrating the algorithm and data structure developed in this paper.

The short-period pitch dynamics of the missile can be described as (see Ref. 10)

$$\dot{\alpha} = Q + \frac{\bar{q}S}{mV_m} C_z(\alpha, M_m, \delta_e)$$

$$\dot{Q} = \frac{\bar{q}S\bar{c}}{I_{yy}} C_m(\alpha, M_m, \delta_e) \qquad (50)$$

The aerodynamic coefficients take the form

$$C_z = C_{z0}(\alpha, M_m) + C_{ze}\delta_e$$

$$C_m = C_{m0}(\alpha, M_m) + C_{me}\delta_e$$

where

$$C_{z0}(\alpha, M_m) = -0.5052\alpha + 0.0429 + (0.1230\alpha - 0.0191)M_m$$

$$C_{m0}(\alpha, M_m) = -0.0055\alpha^3 + 0.2131\alpha^2 - 2.7419\alpha - 0.0381 + (0.0014\alpha^3 - 0.0623\alpha^2 + 0.8715\alpha - 0.4041)M_m$$

$$C_{ze}(\alpha) = 0.09, \qquad C_{me}(\alpha) = -0.675$$

Note that these data are fictitious.

The objective of a pitch autopilot is to ensure tracking of the angle-of-attack command subject to the pitch control surface constraints. Therefore, we formulate the nonlinear H_∞ control problem as

$$\dot{\alpha} = Q + \frac{\bar{q}S}{mV_m}[C_{z0}(\alpha, M_m) + C_{ze}\delta_e]$$

$$\dot{Q} = \frac{\bar{q}S\bar{c}}{I_{yy}}[C_{m0}(\alpha, M_m) + C_{me}\delta_e] + w$$

$$\dot{\alpha}_i = \alpha$$

$$z_1 = \rho_1\alpha$$

$$z_2 = \rho_2\delta_e$$

$$z_3 = k_1\alpha + k_2Q + k_3\alpha_i \qquad (51)$$

Here w is used to model the uncertainty in C_{m0}; α_i is introduced to provide integral control; ρ_1 and ρ_2 are penalties on α and δ_e, respectively; and k_1, k_2, and k_3 are used to shape the output response. A similar idea has been used in Ref. 11. Let $x = (\alpha, Q, \alpha_i)$, $z = (z_1, z_2, z_3)$, and $u = \delta_e$. Then Eq. (51) can be cast in the standard form given by Eq. (1) with

$$f(x) = \begin{bmatrix} Q + \dfrac{\bar{q}S}{mV_m}C_{z0}(\alpha, M_m) \\ \dfrac{\bar{q}S\bar{c}}{I_{yy}}C_{m0}(\alpha, M_m) \\ \alpha \end{bmatrix}$$

$$g_1(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} \dfrac{\bar{q}S}{mV_m}C_{ze} \\ \dfrac{\bar{q}S\bar{c}}{I_{yy}}C_{me} \\ 0 \end{bmatrix}$$

$$h_1(x) = \begin{bmatrix} \rho_1\alpha \\ 0 \\ k_1\alpha + k_2Q + k_3\alpha_i \end{bmatrix}, \qquad k_{12} = \begin{bmatrix} 0 \\ \rho_2 \\ 0 \end{bmatrix}$$

Note that all functions but $f(x)$ are linear and $f(x)$ is a polynomial of degree 3 and can be expressed as

$$f(x) = Ax + A_2x^{[2]} + A_3x^{[3]}$$

We will consider finding a third-order approximate solution of Eq. (9) as

$$V(x) = \tfrac{1}{2}x^TPx + P_3x^{[3]}$$

where P can be obtained by solving the Riccati equation

$$A^TP + PA + C_1^TC_1 - P^T\left[B_2R_2^{-1}B_2^T - (1/\gamma^2)B^1B_1^T\right]P = 0$$

and P_3 can be obtained from solving the linear algebraic equation

$$P_3U_3 = V_3$$

where

$$U_3 = M_3[(H_{px} + H_{pp}P) \otimes I_9 + I_3 \otimes (H_{px} + H_{pp}P) \otimes I_3 + I_9 \otimes (H_{px} + H_{pp}P)]N_3$$

$$V_3 = -\text{row}(S_2A_2)N_3 = -\text{row}(PA_2)N_3$$

Finally, the approximate nonlinear H_∞ control law is given by

$$u = -\left(1/\rho_2^2\right)B_2\left(Px + S_3^TN_3x^{[2]}\right)$$

where

$$S_3 = P_3^1 + P_3^2 + P_3^3 \qquad (52)$$

In Eq. (52), P_i^j is determined by P_3 according to Eq. (32).

The performance of this autopilot is evaluated assuming the missile's initial altitude and velocity are 40,000 ft and 2.75 Mach, respectively. The design parameters are given as

$$\gamma = 0.15, \qquad \rho_1 = 21, \qquad \rho_2 = 7$$

$$k_1 = 32, \qquad k_2 = 1, \qquad k_3 = 520$$

These values are based on a few iterations. The performance of the autopilot is evaluated using computer simulation. Figure 2 shows the profiles of the missile's angle of attack, pitch rate, pitch control, and normal acceleration. The commanded angle of attack is a series of step commands with the magnitudes being 10, 15, and 20 deg, respectively. It is seen that the autopilot exhibits an almost uniform performance in terms of the rise time, overshoot, and settling time with respect to a wide range of flight conditions (from $\alpha = 0$ to 20 deg).

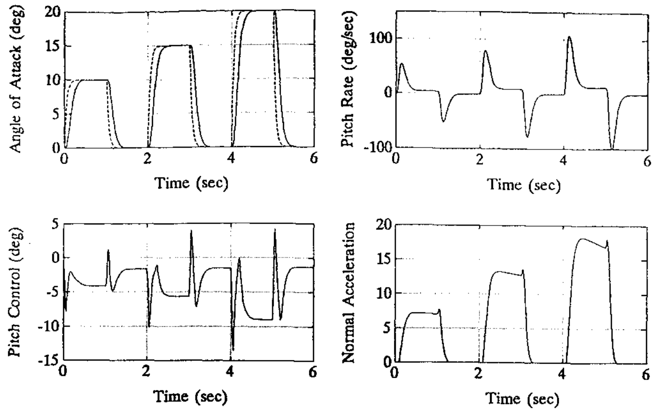

Fig. 2 Performance of nonlinear H_∞ pitch autopilot (dashed line: commanded angle of attack; solid line: angle of attack).

Concluding Remarks

In this paper, we have developed an algorithm and data structure for numerically computing nonlinear H_∞ controllers. This algorithm leads to a systematic Taylor series solution of the Hamilton–Jacobi–Isaacs equation. The algorithm has been applied to the design of a nonlinear H_∞ pitch autopilot. Simulation shows satisfactory results.

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